2019-Sep-10 (lecture)

Tuesday, September 10, 2019 7:35 AM

Prof. Yun William Yu

Outline

1. On proofs & abstraction
   - What did we prove on Thursday?
   - Why is it important?
   - What are the different levels for solving a problem?
2. Explicit Solutions
   - First-order autonomous equations
   - Separable equations
   - ODEs with homogeneous coefficients
   - Exact differentials

What is a mathematical proof?

A mathematical proof is a series of logically connected statements starting from what you know and ending with what you want to prove.

(Thm of Apple deliciousness) **Claim:** Apples are delicious

Proof. It is self-evident that sweet foods are delicious (starting point)

By the theorem of Sugary Fruits, all fruits are sugary foods. (reference to a proven theorem)

The computation below (omitted) shows that apples are fruits. (computation)

Therefore, apples are delicious. (implication) $\square$

- Each step must be justified by an agreed upon reason.
- The amt of justification you need depends on your audience.

Once a claim is rigorously proved, you can reuse it.

Claim: Oranges are delicious

proof: Oranges are apples. ← incorrect starting point

Therefore, oranges are fruits. ← true via the proof of apple deliciousness (wrong b/c of prior step)

By the Thm of Apple Deliciousness, oranges are delicious. ← used wrong theorem

Recall: Consider a first-order autonomous equation.

$$\dot{x} = f(x), \quad f \in C(\mathbb{R})$$

We want to find a function $\phi(t)$ s.t.

$$\dot{x}(\phi(t)) = f(\phi(t))$$

} i.e. find a general solution to ODEs of this type

Ex. $\dot{x} = x^2 + 5x + 10$

$\dot{x} = \ln(x)\, e^{x + \tan(x)}$

Because $\dot{x}$ is time-invariant, we need only consider solutions starting at $x_0 = x(0)$, $t = 0$.

We proved that if $f(x_0) \neq 0$, we can define a function $F(x) = \displaystyle\int_{x_0}^{x} \frac{dy}{f(y)}$ in some interval of validity (x_1, x_2)

where $f(x) \neq 0 \quad \forall\, x \in (x_1, x_2)$

$\Rightarrow F(x)$ is strictly monotonic

Why did we define $F(x)$ and prove strict monotonicity?

Suppose $\phi(t)$ is a solution to $\dot{x} = f(x)$.

Then $$F(\phi(t)) = \int_{x_0}^{t} \frac{\dot{\phi}(s)}{f(\phi(s))}\, ds = \int_0^t ds = t$$

If $F(\phi(t)) = t$, then $\phi(t) = F^{-1}(t)$, where F^{-1} exists

F^{-1} exists as long as $F(x)$ is strictly monotonic

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F^{-1} exists so long as $F(x)$ is strictly monotonic.

This proves that we can find a solution by integrating, if $f(x_0) \neq 0$.

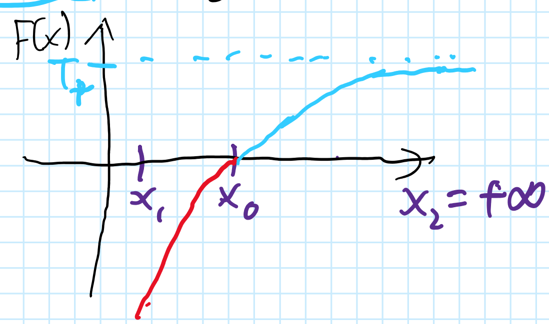
But these solutions are only valid on (x_1, x_2).

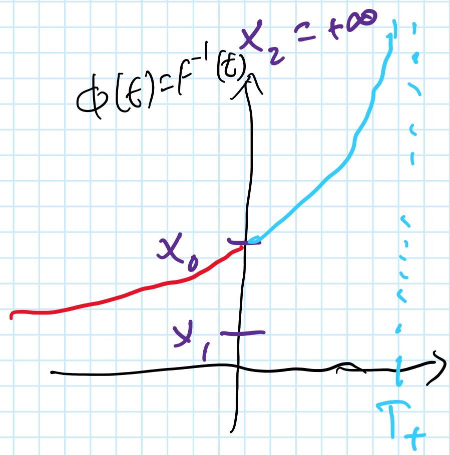
What happens as we get closer to those limits?

Let's consider x_2 for $F(x)$ strictly positive monotonic

Recall $T_+ = \lim_{x \uparrow x_2} F(x) \quad \left(= \lim_{x \to x_2^-} F(x) \right)$

Case 1: $x_2 = +\infty$

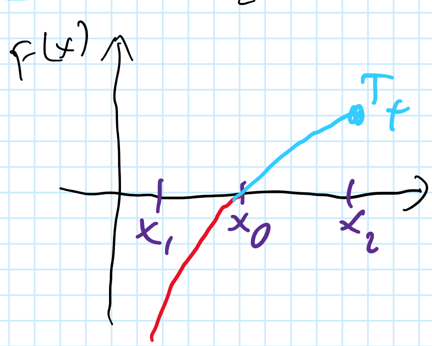


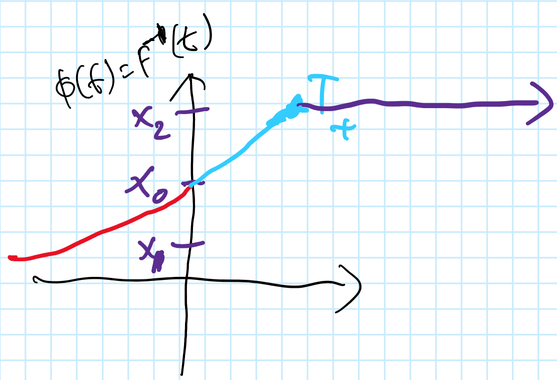


$\phi(t)$ asymptotically approaches $+\infty$, as $t \uparrow T_+$,

so this function cannot be continuously extended

Case 2: $x_2 < +\infty$





$\phi(t)$ approaches a finite T_+, and $f(x_2) = 0$.

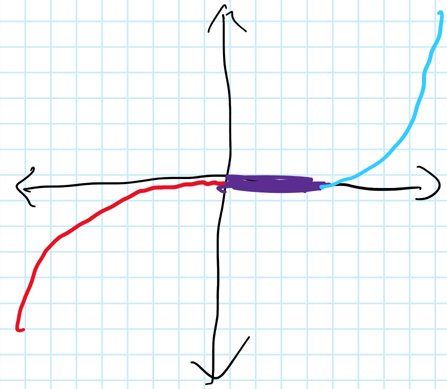
So we can extend the constant solution.

Odd behavior can appear when $f(x_0) = 0$

Ex. $f(x) = \sqrt{|x|}$, $x_0 = 0$

$$\Rightarrow \phi(t) = \begin{cases} -\frac{(t-t_0)^2}{4}, & t \le t_0 \\ 0, & t_0 \le t \le t_1 \\ \frac{(t-t_1)^2}{4}, & t_1 \le t \end{cases}$$

(Ex. 1.33 Teschl)



Conclusion: We can usually integrate to get solutions for 1st order autonomous ODEs.

But, solutions might only exist locally in t.

And, solutions might not be unique

Levels of Abstraction

More abstract/general

$F(t, x, x', x'', \ldots, x^{(k)}) = 0$ (general ODE)

$x^{(k)} = f(t, x, x', \ldots, x^{(k-1)})$ (explicit form for kth-order)

first order $(x' = f(t,x))$, second order $(x'' = f(t, x, x'))$, ...

first order autonomous $(\dot{x} = f(x))$, separable $(\dot{x} = g(t) f(x))$, ...

$\dot{x} = -kx$, $\dot{x} = ax + b$, $\dot{x} = c_1 \sin(x) + c_2 \cos(x)$, ...

$\dot{x} = -5x$, $\dot{x} = 5x + 2$, ...

radiocarbon dating, compound interest, falling objects

More specific/applied

Explicit Solutions: We just gave a general solution for one class of ODEs. This and other general

Explicit Solutions: We just gave a general solution for one class of ODEs. This and other general solutions allow us to solve specific problems.

Example $\dot{x} = x^2, \quad x(0) = x_0$

$$\frac{dx}{dt} = x^2$$

If $x_0 \neq 0$, $\quad \dfrac{dx}{x^2} = dt$

$$\int \frac{dx}{x^2} = \int dt + C$$

$$-\frac{1}{x} = t + C$$

$$\frac{1}{x} = -t + C$$

$$x = \frac{1}{-t + C}$$

$$x(0) = \frac{1}{0 + C} = x_0$$

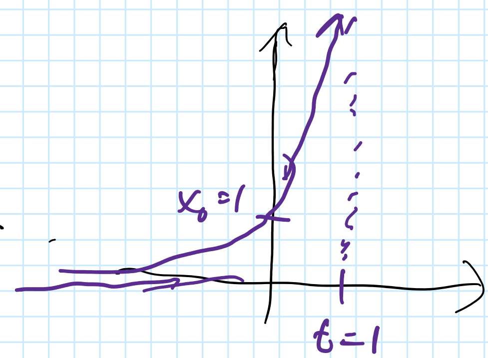
$$\frac{1}{C} = x_0$$

$$C = \frac{1}{x_0}$$

$$x = \frac{1}{-t + \frac{1}{x_0}} = \frac{x_0}{1 - x_0 t}$$

Include explicit transforms for constants

$$\left(\begin{aligned} -\frac{1}{x} &= t + C_1 \\ \frac{1}{x} &= -t + C_2, \quad C_2 = -C_1 \\ x &= \frac{1}{-t + C_2} \end{aligned} \right)$$



If $x_0 = 0$, and you plugged it in

$$x = \frac{0}{1 - 0} = 0.$$

This is a solution, though not one given by our solving method. (since $x_0 = 0$ is not allowed in division)

Note: Whenever we do integration, we often end up with not just one solution, but an entire family of solutions because of constants.

Define A k-parameter family of solutions for an ODE to be a solution that has k constant terms $C_1, \dots, C_k$

Ex. $f''''(t) = 5$

$$f'''(t) = 5t + C_1$$
$$f''(t) = \frac{5}{2}t^2 + C_1 t + C_2$$
$$f'(t) = \frac{5}{6}t^3 + \frac{1}{2}C_1 t^2 + C_2 t + C_3$$
$$f(t) = \frac{5}{24}t^4 + \frac{1}{6}C_1 t^3 + \frac{1}{2}C_2 t^2 + C_3 t + C_4$$

4-parameter family

Ex. (from above) $\dot{x}(t) = x^2(t)$

$$x(t) = \frac{1}{-t+C} \leftarrow \text{1-parameter family}$$

$$x(t) = \frac{x_0}{1 - x_0 t} \leftarrow \text{1-parameter family with constant } x_0$$

Ex. $y'' - y = 0$

We haven't learned how to solve this yet, but you plug & check.

$$\Rightarrow y = C_1 e^x + C_2 e^{-x}$$

2 parameter family

$$y' = C_1 e^x - C_2 e^{-x}$$
$$y'' = C_1 e^x + C_2 e^{-x} = y$$

For many (but not all) ODEs, we can say that a kth-order ODE has a k-parameter family of solutions

Def. A particular solution is a solution that has no arbitrary constants

Def. A general solution is a k-parameter family of

Def. A general solution is a k-parameter family of solutions that contains every particular solution.
(Tenenbaum)

Note: I will sometimes use general to mean the collection of all solutions

Using this terminology, we say that we can find a 1-parameter family of solutions to a 1st-order autonomous ODE by integrating. If $f(x) \neq 0$, $\dot{x} = f(x)$ anywhere in the domain, that 1-parameter family of solutions is general.

Other types of ODEs

Separable coefficients

Define: A separable first-order ODE is an ODE that can be rewritten

$$\dot{x}(t,x) = g(t) f(x).$$

(or equivalently, $g(t)dt + f(x)dx = 0$)

We can solve separable equations like we solved 1st order autonomous using the same ruse.

$$\frac{dx}{dt} = g(t) f(x)$$

$$\frac{dx}{f(x)} = g(t)\,dt$$

$$\int \frac{dx}{f(x)} = \int g(t)\,dt \text{ , so long as } f(x) \neq 0.$$

Ex. $\dot{x}(t,x) = \sin(t) \cdot (x+1)$

$$\frac{dx}{dt} = \sin(t) \cdot (x+1)$$

$$\frac{dx}{dt} = \sin(t) \cdot (x+1)$$

$$\frac{dx}{x+1} = \sin(t)\, dt$$

$$\int \frac{dx}{x+1} = \int \sin(t)\, dt$$

If $x > -1$, $\ln(x+1) = -\cos(t) + C$

$$x+1 = e^{-\cos(t)+C}$$

$$x+1 = Ce^{-\cos(t)}, \quad C > 0$$

$$x = -1 + Ce^{-\cos(t)}, \quad C > 0$$

If $x < -1$, $\ln(-x-1) = -\cos(t) + C$

$$-x-1 = e^{-\cos(t)+C}$$

$$-x-1 = Ce^{-\cos(t)}, \quad C > 0$$

$$x+1 = -Ce^{-\cos(t)}, \quad C > 0$$

$$x = -1 - Ce^{-\cos(t)}, \quad C > 0$$

If $x_0 = -1$, $x = -1$.

So now we can rewrite as a single general solution

$$x = -1 + Ce^{-\cos(t)}, \quad C \in \mathbb{R}$$

1-parameter family

Ex 6.7 Find a particular solution of $xy^2 dx + (1-x) dy = 0$ for which $y(2) = 1$.

Tenenbaum

Let's rewrite this in separated form

$$xy^2 dx + (1-x) dy = 0$$

Divide by $y(1-x)$:
$$\frac{x}{1-x} dx + \frac{1}{y^2} dy = 0$$

$$\int \frac{x}{1-x} dx + \int \frac{1}{y^2} dy = C$$

$$\int\left(\frac{1}{1-x}-1\right)dx+\int\frac{1}{y^2}dy=C$$

$$\int\frac{1}{1-x}dx-\int 1\,dx+\int\frac{1}{y^2}dy=C$$

$$-\ln|1-x|-x-\frac{1}{y}=C$$

↑ 1 parameter family of solutions (implicit)

Recall $y(2)=1$ (i.e. $x=2$, $y=1$ is a point on the curve of the particular solution)

$$-\ln|1-2|-2-1=C$$

$$0-2-1=C$$

$$C=-3$$

Particular soln: $-\ln|1-x|-x-\frac{1}{y}=-3$

$$\ln|1-x|+x+\frac{1}{y}=3$$

Homogeneous coefficients - do NOT confuse with homogeneous linear ODEs which don't have a $g(t)$ term.

Def. Let $z=f(x,y)$ be a function of x and y.
$f(x,y)$ is homogeneous of order n if it can be written as $f(x,y)=x^n g(u)$, where $u=\frac{y}{x}$

or $f(x,y)=y^n g(u)$, where $u=\frac{x}{y}$

Ex. $f(x,y)=x^2+y^2\log\frac{y}{x}$, $x>0$, $y>0$.
$f(x,y)=x^2\left(1+\frac{y^2}{\ldots}\ldots\frac{y}{\ldots}\right)$

Ex. $f(x,y) = x^2 + y^2 \log \frac{y}{x}$, $x > 0$, $y > 0$.

$$f(x,y) = x^2\left(1 + \frac{y^2}{x^2}\log\frac{y}{x}\right)$$

Substituting in $u = \frac{y}{x}$, $f(x,y) = x^2(1 + u^2 \log u)$

Thus $x^2 + y^2 \log \frac{y}{x}$ is homogeneous of order 2.

Alternately, $f(x,y) = y^2\left(\frac{x^2}{y^2} + \log\frac{y}{x}\right)$

$$= y^2\left(u^2 + \log\frac{1}{u}\right) = y^2(u^2 - \log u)$$

still homogeneous of order 2.

Alternately, an equivalent definition is that a function $f(x,y)$ is **homogeneous of order n** if

$$f(tx, ty) = t^n f(x,y)$$

Ex. $f(x,y) = x^2 + y^2 \log \frac{y}{x}$

$$f(tx, ty) = t^2x^2 + t^2y^2 \log\frac{ty}{tx} = t^2\left(x^2 + y^2 \log\frac{y}{x}\right)$$

Are the following homogeneous? What order?

1. $e^{y/x} + \tan\left(\frac{y}{x}\right)$. Set $u = \frac{y}{x}$. $\Rightarrow (e^u + \tan u)\,x^0$

   homogeneous order 0

2. $x^2 + \sin x \cos y$ — not homogeneous — substitute tx, ty: $t^2x^2 + \sin tx \cos ty$

3. $\sqrt{x+y}$ — order $\frac{1}{2}$. $\sqrt{tx+ty} = t^{\frac{1}{2}}\sqrt{x+y}$

4. $\sqrt{x^2 + 3xy + 2y^2}$ — order 1.

5. $x^4 - 3x^3y + 5y^2x^2 - 2y^4$ — order 4

Theorem 7.32 (Tenenbaum) Given an ODE $P(x,y)\,dx + Q(x,y)\,dy = 0$, where $P(x,y)$ and $Q(x,y)$ are homogeneous with the same order, the substitution

$y = ux, \quad dy = u\,dx + x\,du$

leads to a separable ODE in u and x.

Ex. $2ye^{x/y}\,dx + (y - 2xe^{x/y})\,dy = 0$

Note $2ye^{x/y}$ is homogeneous with order 1

$y - 2xe^{x/y}$ is " " " " " .

Let $x = uy \quad dx = u\,dy + y\,du$

$$2ye^{u}(u\,dy + y\,du) + (y - 2uye^{u})\,dy = 0$$

$$2uye^{u}\,dy + 2y^2e^{u}\,du + y\,dy - 2uye^{u}\,dy = 0$$

$$2y^2e^{u}\,du + y\,dy = 0$$

$$2e^{u}\,du + \frac{1}{y}\,dy = 0$$

$$2e^{u} + \ln|y| = C$$

$$2e^{x/y} + \ln|y| = C.$$